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# PICARD'S EXISTENCE AND UNIQUENESS THEOREM

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**Abstract**

***The study of fixed point theory in differential equation has mainly developed in generalization of conditions which ensure existence and if possible, uniqueness of a solution. In this paper we have discussed the necessary and sufficient condition for the function for the existence of a solution of differential equation.***

***Keywords: Picard's theorem, Lipschitz condition, continuity, Banach Fixed point theorem***

## I. INTRODUCTION

The study of fixed point theory has been researched widely in the last few decades. Fixed point theory has been applied to various fields of mathematics and plays a major role in uniqueness and existence of a solution.

In mathematics, in the area of differential equations, Cauchy – Lipchitz theorem, the Picard – Lindelof theorem, Picard's Existence theorems are important theorems on existence and uniqueness of solutions to differential equation with initial conditions.

## II. PRELIMINARIES

1. **Banach Fixed point theorem:** " Consider a complete metric space $X = (X, d)$, and let $T: X \rightarrow X$ be a contraction mapping on X then there exist one and only one fixed point $x \in X$ such that $Tx = x$"

**Proof**: Let $x, y \in X$, then $d[f(x), f(y)] \leq \alpha d(x, y)$, Since f is contraction on X

$$d[f^2(x), f^2(y)] \leq \alpha d[f(x), f(y)] \leq \alpha^2 d(x, y)$$

$$\vdots$$

$$d[f^n(x), f^n(y)] \leq \alpha^n d(x, y) \qquad \text{for } \forall x, y \in X. \qquad (by\ induction)$$

$Now\ let\ x_0 \in X, then \quad x_1 = f(x_0)$

$$x_2 = f(x_1) = f^2(x_0)$$

$$x_3 = f(x_2) = f^2(x_1) = f^3(x_0)$$

$$\vdots$$

$x_n = f^n(x_0) \quad \Rightarrow \{x_n\}\ is\ a\ Cauchy'sequence$

For Cauchy's sequence, let $m > n, m = n + p, for\ some\ p \geq 1$

$$d(x_n, x_m) = d(x_n, x_{n+p})$$

$$\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots\cdots + d(x_{n+p-1}, x_{n+p})$$

$$\leq \alpha^n d(x_0, x_1) + \alpha^{n+1} d(x_0, x_1) + \cdots\cdots + \alpha^{n+p-1} d(x_0, x_1)$$

 

$\leq \alpha^n d(x_0, x_1)[1 + \alpha + \alpha^2 + \alpha^3 + \cdots\cdots + \alpha^{p-1}]$
$\leq \alpha^n \frac{d(x_0,x_1)}{[1-\alpha]} \to 0, as\ n \to \infty$

$\Rightarrow \{x_n\}\ is\ a\ Cauchy'sequence$
$but\ X\ is\ complete\ \Rightarrow \exists x \in X\ s.t.\ x_n \to x\ as\ n \to \infty$
$\Rightarrow \lim_{n\to\infty} f(x_n) = f(x) \ldots\ldots (a)$
$\Rightarrow \{f(x_n)\}\ is\ subsequence\ of\ \{x_n\}, hence\ converges\ to\ x$
$\therefore \lim_{n\to\infty} f(x_n) = x \ldots\ldots\ldots (b)$
From (a) and (b), $f(x) = x, \Rightarrow x\ is\ a\ fixed\ point \ldots\ldots \{proves\ existance\}$
Uniqueness: let $\in X, be\ another\ fixed\ point\ then\ f(y) = y$
$d(x,y) = d[f(x), f(y)] \leq \alpha d(x,y),$
$\because d(x,y) \neq 0, \quad \alpha \geq 1\ which\ is\ contradiction$
$\therefore f(y) \neq y, \qquad hence\ the\ uniqueness.$

2. **Lipschitz condition:** A function f from X to Y is Lipschitz continuous at x ,y ∈ X if there is a constant k such that | f(y) − f(x)| ≤ k |y – x|
   Any such k is referred to as a Lipschitz constant for the function f.
3. A metric M is called Complete if every Cauchy sequence in M has a limit in N, that is Every Cauchy sequence in M converges in M

## III. THEOREM: PICARD'S EXISTENCE AND UNIQUENESS THEOREM

Statement: Let f be Lipschitz continuous on R,
$R = \{(t,x)|, |t - t_0| \leq a, |x - x_0| \leq b\},$
Then $\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0$ ……….(1),
$f(t,x) \leq c, \forall t, x \in R$ ………… (2)
has unique solution in $[t_0 - \propto, t_0 + \propto]$, $\propto < \min\{a, \frac{b}{c}, \frac{1}{k}\}$ …………(3)
***Proof***: Since f is Lipschitz continuous, for some $(t,x), (t,u) \in R$
$|f(t,x) - f(t,u)| \leq k|x - u|$ ……….. (4)
Let (C, d) be the metric of real valued continuous function on $[t_0 - \propto, t_0 + \propto]$, such that
d(x, y) = Max |x(t) – y(t) |
Then C is complete
Let C' be the subspace of C satisfying
$|x(t) - x_0| \leq c \propto, for\ all\ x \in C$ ……………..(5)
$\therefore C' is\ closed\ and\ hence\ complete.$
Define $T: C' \to C' as\ T(x(t)) = x_0 + \int_{t_0}^{t} f(v, x(v))dv,$ ……………(6)
$for\ x \in C', v \in [t_0 - \propto, t_0 + \propto], (\text{v}, \text{x}(\text{v})) \in R.$
Now from (2) and (6)

$$|T(x(t)) - x_0| = |\int_{t_0}^{t} f(v, x(v))dv|$$
$$\leq c|t - t_0|$$
$$\leq c \propto [since\ c \propto < b\ from\ (3)]$$

$\therefore T\ maps\ C' into\ itself.$
Now by (4)

$$|T(x(t)) - T(u(t))| = \left|\int_{t_0}^{t} [f(v, x(v)) - f(v, u(v))]dv\right|$$

$$\leq |t - t_0| \max k|x(v) - u(v)|$$
$$\leq k \propto d(x,u)$$

$\therefore d(Tx, Tu) \leq \beta d(x,u),\ \ where\ \beta = \propto k$

From (3), $\beta < 1 \quad \Rightarrow T\ is\ contraction\ in\ C'$,

Then by Banach Fixed point theorem,

T has unique fixed point $x \in C'\ such\ that\ Tx = x$

$$\therefore from\ (6), \qquad x(t) = x_0 + \int_{t_0}^{t} f\big(v, x(v)\big)dv, \qquad \ldots\ldots\ldots\ldots\ldots (7)$$

Where *x(t)* is differentiable and satisfies (1), hence the proof.

The condition on function $f$ being continuous is sufficient but not necessary for existence of a solution of (1) whereas it can be easily seen that it is not sufficient for uniqueness of a solution.

For instance, $Let\ \frac{dy}{dx} = y(1-2x), x > 0$

$$= y(2x-1), x < 0$$

Where f is continuous and satisfies initial condition y (1) = 1

Whose solution is $y = e^{x-x^2}, x \geq 0$

$$= e^{x^2-x}, x \leq 0 \quad \text{for all real } x$$

Which is unique continuous solution.

A Lipchitz condition should be imposed to confirm the uniqueness. Otherwise there exists more than one continuous solution.

Further consider the differential equation

$$f(x,u) = \frac{4x^3u}{x^4+u^2}, for\ non\ zero\ x, u$$
$$= 0, \qquad for\ x = y = 0.$$

Now $f(x,u) - f(x,v) = \frac{4x^3u}{x^4+u^2} - \frac{4x^3v}{x^4+v^2}$

$$= \frac{4x^3(x^4-uv)(u-v)}{(x^4+u^2)(x^4+v^2)}$$

$$|f(x,u) - f(x,v)| = 4\left|\frac{x^3(x^4-uv)(u-v)}{(x^4+u^2)(x^4+v^2)}\right|$$

This is violation of Lipschitz condition.

This equation has a solution $y = C - \sqrt{x^4 + C^4}$, since C is arbitrary constant, it has many solutions satisfying the initial condition *x = 0, y = 0*

## IV. CONCLUSION

Thus it can be concluded that the continuity of a function is sufficient to have a solution of a differential equation whereas for uniqueness it is not sufficient. Additionally Lipschitz condition to be satisfied to have a unique solution.

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